Journal of Sound and Vibration (1999) **219**(5), 920–924
Article No. jsvi.1998.1913, available online at http://www.idealibrary.com on IDEAL

VIBRATION MODES IN A STRING OF THREE BRANCHES

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(*Received* 4 *March* 1998, *and in final form* 31 *July* 1998)

1. INTRODUCTION

The classic problem of the transverse vibrations of a string is of basic and technological interest, since it constitutes an acceptable model for the dynamic behavior of oceanographic cables and musical instruments among others [1, 2]. One of the important points in this area is the propagation of waves in systems with interferences or in mediums with different propagation speed. However, the literature shows, in general, very little information on the main aspects of stationary waves in these types of systems. In this work we studied the behavior of the transversal stationary waves in a mechanical system composed of three strings connected to a ramification point (Figure 1). First, the spectrum of natural frequencies of the system for a completely general case is presented. Then two particularly simple situations are studied: the case of a string with three identical branches and that of a string with two equal branches and the third of different length. Results obtained for a string with branches of different chain lengths are similar to those for a linear string of fixed ends in some particular situations.

1.1. *Equation of movement for a branched string*

In order to calculate the frequency spectrum of a branched string, a separable one-dimensional wave equation is considered for each of the branches of the string:

$$\frac{\partial^2 u^i(x_i, t)}{\partial t^2} = \delta_i^2 \frac{\partial^2 u^i(x_i, t)}{\partial x_i^2}, \tag{1}$$

where $u^i(x_i, t)$ represents the position as a function of time and co-ordinate x_i for the i-branch and δ_i is given by the relationship:

$$\delta_i^2 = \tau_i / \mu_i, \tag{2}$$

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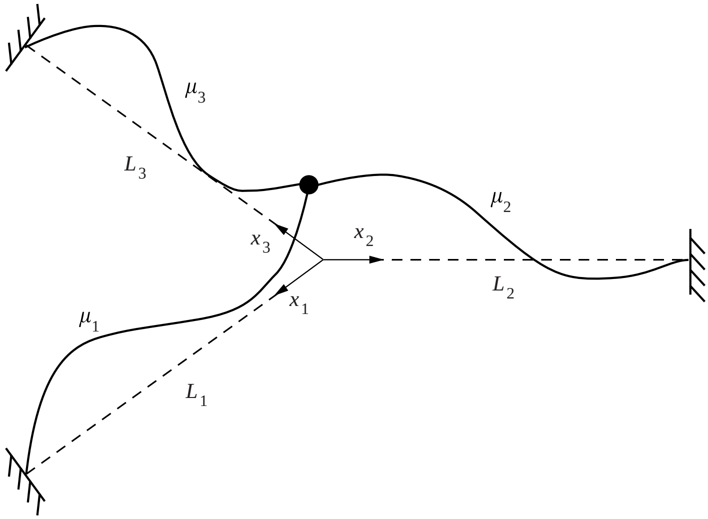

Figure 1. String of three branches fixed at their ends.

where μ_i is the mass by unit of length and τ_i is the tension in each branch. The tensions τ_i are related by a static equilibrium condition.

In order to get the normal modes of vibration, we have the condition

$$\omega^2 = \frac{\tau_1}{\mu_1} k_1^2 = \frac{\tau_2}{\mu_2} k_2^2 = \frac{\tau_3}{\mu_3} k_3^2, \tag{3}$$

In this work, a system is analyzed where the lengths of each string are L_1, L_2 and L_3 and the positions $u^i(x_i, t)$ are zero at the fixed ends of the branches. The boundary conditions to satisfy the system at the ramification point are the conditions of continuity for the position and transversal stress for the branched point [3, 4]. This last condition can be written as

$$\tau_1 \frac{\partial u^1(x_1, t)}{\partial x_1} + \tau_2 \frac{\partial u^2(x_2, t)}{\partial x_2} + \tau_3 \frac{\partial u^3(x_3, t)}{\partial x_3} = 0 \qquad \text{at } x_1 = x_2 = x_3 = 0. \tag{4}$$

Then, to obtain a solution different from the trivial one, the following equation must be satisfied:

$$\sin(k_1 L_1)[\tau_2 \sin(k_3 L_3)\cos(k_2 L_2) + \tau_3 \sin(k_2 L_2)\cos(k_3 L_3)] + \tau_1 \cos(k_1 L_1)\sin(k_2 L_2)\sin(k_3 L_3) = 0. \tag{5}$$

The wave numbers allowed for each branch and the spectrum of natural frequencies can be determined from equations (5) and (3).

Now two particularly simple cases are analyzed where the stress and the masses by unit of length are assumed the same for each branch ($\mu_i = \mu$, $\tau_i = \tau$, $i = 1, 2, 3$). For this configuration, it can be shown that the angle between each branch of the strings should be 120°.

First, the spectrum of natural frequencies for a string of three branches with identical length are analyzed and then the spectrum corresponding to a string composed of two branches with the same length and one of different length are presented.

1.1.1. *Three branches identical*

If the three branches of the string have the same length L, then equation (5) is reduced to

$$\sin^2(Lk)\cos(Lk) = 0. \tag{6}$$

The solutions of this equation are

$$k_p = \frac{p\pi}{L}, \qquad p = 1, 2 \ldots, \tag{7}$$

and

$$k_q = \frac{(2q-1)\pi}{2L}, \qquad q = 1, 2 \ldots. \tag{8}$$

For the values of k_p that satisfy equation (7) the eigenfunctions for $x = 0$ are proportional to $\sin(p\pi) = 0$ for $p = 0, 1 \ldots$; they have nodes at the branching point. On the other hand, for the values of k_p that satisfy equation (8) the eigenfunctions have a crest at $x = 0$.

The eigenfrequencies of the system can be obtained with equations (3), (7) and (8):

$$\omega_n = \sqrt{\frac{T}{\mu}}\left[\frac{n\pi}{2L}\right], \qquad n = 1, 2 \ldots. \tag{9}$$

Then, a string with three identical branches has the same spectrum as a linear string [5] with fixed ends and length $2L$.

1.1.2. *Two branches with the same length*

Finally, the solution of a string composed of two branches of the same length L and a third branch with length L_2 is examined. In this case equation (5) is reduced to the relationship

$$\sin(kL)[\sin(kL)\cos(kL_2) + 2\sin(kL_2)\cos(kL)] = 0. \tag{10}$$

For this equation the solutions are

$$k_p = \frac{p\pi}{L}, \qquad p = 1, 2 \ldots, \tag{11}$$

and

$$\tan(k_q L) = -2\tan(k_q L_2), \qquad q = 1, 2 \ldots. \tag{12}$$

Equation (11) gives the frequencies for those modes where the eigenfunctions have a node at the branching point.

TABLE 1

Values of frequency coefficient Ω_p as function of L_2 for a branched string ($L = 1$)

	L_2						
Mode (p)	0·01	0·1	0·5	1	2	10	100
1	3·141 ($\approx \pi/L$)	2·832	2·528	2·204	1·320	0·299	0·031 ($\approx \pi/L_2$)
2	6·283 ($\approx 2\pi/L$)	3·141	3·141	3·141	2·423	0·600	0·062 ($\approx 2\pi/L_2$)
3	9·424 ($\approx 3\pi/L$)	6·021	5·636	5·806	3·141	0·904	0·093 ($\approx 3\pi/L_2$)

If $L_2 \gg L$ then the first roots of equation (12) will be approximately the zeros of tan $(k_q L_2)$:

$$k_q \approx \frac{q\pi}{L_2}. \tag{13}$$

In this particular limit ($L_2 \gg L$) the system will show at low frequencies, similar characteristic frequencies to those of a single string of length L_2 with fixed ends.

On the other hand, if $L_2 \ll L$, the first solutions of equation (12) are those corresponding to tan $(k_q L)$ equal to zero. These solutions are equivalent to those obtained by equation (11). Then, for $L_2 \ll L$, the branched string behaves, at low frequencies, like two independent linear strings of length L.

Table 1 shows the values for the frequency coefficient $\Omega_p = (\mu/\tau)^{1/2}\omega_p$ for a branched string with two branches of the same length L and the third branch with length L_2. In this table the frequency coefficient for the three first modes are evaluated for $L = 1$ and L_2 varying between 0·01 and 100.

For $L_2 = 100$ (where the condition of $L_2 \gg L$ is satisfied) calculated values of Ω_p are very similar to those corresponding to a string of length L_2 with fixed ends. On the other hand, if $L_2 = 0{\cdot}01$ ($L_2 \ll L$) one obtains the same spectrum as that for a linear string of length L, which is given by

$$\omega_n = \sqrt{\frac{T}{\mu}}\left[\frac{n\pi}{L}\right], \qquad n = 1, 2 \ldots. \tag{14}$$

The analysis presented here is also applicable to the study of stationary acoustical or electromagnetic waves in systems where the problem can be described by a unidimensional wave equation for each branch. In this work, only stationary waves have been analyzed but the problem can be generalized to the study of wave propagation [6].

ACKNOWLEDGMENT

We would like to thank Dr P. A. A. Laura for his useful contribution to this work. We also express our gratitude to the National Research Council of Argentina (CONICET) and the Universidad Nacional del Sur (UNS) which supported this work.

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